On the Different Shapes Arising in a Family of Rational Curves Depending on a Parameter

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Abstract

Given a family of rational curves depending on a real parameter, defined by its parametric equations, we provide an algorithm to compute a finite partition of the parameter space ($\mathbb{R}$, in general) so that the shape of the family stays invariant along each element of the partition. So, from this partition the topology types in the family can be determined. The algorithm is based on a geometric interpretation of previous work ([1]) for the implicit case. However, in our case the algorithm works directly with the parametrization of the family, and the implicit equation does not need to be computed. Timings comparing the algorithm in the implicit and the parametric cases are given; these timings show that the parametric algorithm developed here provides in general better results than the known algorithm for the implicit case.

1 Introduction

Given a family $\mathcal{F}$ of algebraic curves depending on a real parameter, it is clear that the shape of the curves in the family may change as the value of the parameter changes. In the C.A.G.D. context, one has a good example of this phenomenon in the family of *offset* curves to a given curve (see [3], [11], [14], [15]), where the parameter is the offsetting distance. Take for example the well-known case of the offset to the parabola (see [6]). In this case three different shapes, which can be seen in Figure 1 (together with the original parabola, in thinner line), arise; here, one may observe that for distances $d < 1/2$, the offsets show no cusps, but they exhibit an isolated point (i.e. a *geometric extraneous component*), and that for $d > 1/2$, two cusps and a self-intersection arise.
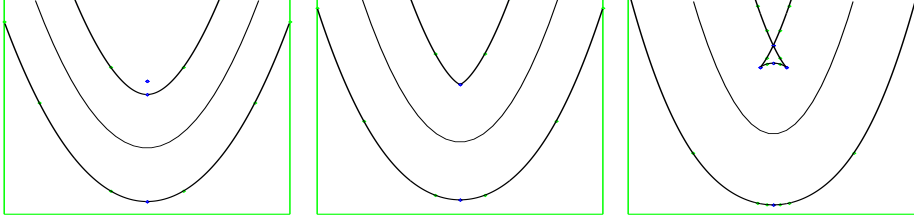

Fig. 1. Topology types of the offsets to $y^2 - x = 0$: $d < 1/2$ (left); $d = 1/2$ (center); $d > 1/2$ (right)
. In the middle of each figure, the parabola has been also plotted.

The problem of computing the different shapes arising in a family of algebraic curves depending on a parameter has been considered in [1], [12], under different perspectives but reaching similar results, for the case when the family is implicitly given; so, in these papers the family to be analyzed is defined as the zero set of a polynomial $F(x, y, \lambda)$, where λ is regarded as a real parameter. In particular, using the result in [1], the topology types in Figure 1 can be determined. For this purpose, the strategy is the following. First, one computes a finite set $\mathcal{A}$ so that in between two consecutive elements of $\mathcal{A}$, the topology type of the family does not change; we refer to such a set as a *critical set*; in the case of the offset to the parabola, $\mathcal{A} = \{1/2\}$. Then, this critical set induces a finite partition of the parameter space; in the above example, the partition is $(0, 1/2)$, $\{1/2\}$, $(1/2, \infty)$ (notice that the offsetting distance is a positive real number). Finally, taking a representative for each element of the partition, and applying if necessary standard methods for describing the topology of a plane algebraic curve (see [5], [8], [9]), the different shapes in the family can be computed.

In this paper, starting from the results in [1] we address the same problem but for the case, specially interesting in the C.A.G.D. context, of a family $\mathcal{F}$ of rational curves depending on a parameter, defined in parametric form. Thus, our input is

$$
\begin{cases}
x = u(t, \lambda) \\
y = v(t, \lambda)
\end{cases}
$$

where $u(t, \lambda), v(t, \lambda)$ are rational functions in terms of t, λ, and $\lambda \in \mathbb{R}$ is a parameter. In these conditions, we consider the computation of a critical set of $\mathcal{F}$, but without computing or making use of the implicit equation of the family. So, the main result of the paper is an algorithm for carrying out this computation.

The algorithm is advantageous when a rational parametrization of the family to be analyzed is available. In fact, in our experimentation we have found

several examples (see the comparison table in Subsection 4.3) of families that are hard to study with the result in [1] (i.e. the computations are too heavy in implicit form), but which can be analyzed with the algorithm provided here.

The algorithm is based on a geometrical interpretation of the results in [1]. This interpretation suggests a certain geometrical process to be performed in order to compute a critical set. Hence, the question is to carry out that process in parametric form. The geometric analysis of the main results in [1] can be found in Section 2. In Section 3, we show how to check different hypotheses that we impose on the parametrization, and how to "prepare" the family before applying the algorithm; the ideas in this section are illustrated in Example 1, where the offset family to a cardioid is considered. In Section 4, we provide the full algorithm, we complete the analysis of Example 1, and we give a comparison table showing the timings of our algorithm, compared to those corresponding to the algorithm in [1]; these timings show that our algorithm yields in general better results than the known algorithm for the implicit case. Finally, in Section 5 we present the conclusions of our study. The parametrizations used for comparing timings are given in Appendix I.

2   Preliminaries.

2.1   Known Results for the Implicit Case.

Let $F \in \mathbb{R}[x, y, \lambda]$. For all $\lambda_0 \in \mathbb{R}$ such that $F(x, y, \lambda_0)$ is not identically 0 we have that $F(x, y, \lambda_0) = F_{\lambda_0}(x, y)$ defines an algebraic curve. So, we can say that F defines a family $\mathcal{F}$ of algebraic curves algebraically depending on the parameter λ. By Hardt's Semialgebraic Triviality Theorem (see Theorem 5.46 in [4]), it holds that the number of topology types of $\mathcal{F}$ (i.e. the different shapes arising in the family as λ_0 moves in $\mathbb{R}$) must be finite. Therefore, it makes sense to consider the problem of computing the topology types arising in the family. This problem has been addressed in [1], for the case when the family is implicitly defined. So, in this subsection we recall the main aspects (hypotheses, notation, and results) of this paper.

More precisely, given $F(x, y, \lambda)$ one can associate an algebraic surface S with the family $\mathcal{F}$ defined by F by substituting $\lambda := z$ in F; thus, the members of $\mathcal{F}$ are the *level curves* of S, i.e. the sections of S with planes normal to the z-axis. So, the problem of computing the topology types in $\mathcal{F}$ can be re-interpreted as the computation of the topology types arising in the family of level curves of S. This is exactly the question addressed in [1]. In order to solve this problem, in [1] the following definition is introduced.

Definition 1 *Let S be an algebraic surface. We say that $\mathcal{A} \subset \mathbb{R}$ is a* critical set *of S, if it is finite and it contains all the z-values where the topology type of the level curves of S changes; i.e., if the topology type of the level curves of S stays invariant along any interval delineated by two consecutive elements of $\mathcal{A}$.*

We speak about the critical set of $\mathcal{F}$, to mean the critical set of the surface associated with the family. Notice that if a critical set $\mathcal{A} = \{a_1, \ldots, a_r\}$ is computed, then the parameter space ($\mathbb{R}$, in this case) can be decomposed as

$$(-\infty, a_1) \cup \{a_1\} \cup (a_1, a_2) \cup \cdots \cup \{a_r\} \cup (a_r, +\infty)$$

Then, taking a representative for each element of the above partition, and applying standard methods ([5], [8], [9]) for describing the topology of an algebraic curve, the topology types in the family can be computed. Hence, the crucial question is the computation of a critical set. This is the problem addressed in [1]. In the rest of the subsection, we recall the hypotheses, notation (that we will also follow here) and main result of [1].

Hypotheses: The hypotheses imposed in [1] on the surface S to be analyzed, are: (a) F is square-free, and depends on the variable y; (b) F does not contain any factor only depending on the variable z (i.e. S has no component normal to the z-axis); (c) the leading coefficient of F w.r.t. the variable y, does not depend on x.

Notation: Given a polynomial G, $\sqrt{G}$ denotes the square-free part of G, i.e. the product of all the irreducible factors of G taken with multiplicity 1. Also, $D_w(G)$ denotes the *discriminant* of G with respect to the variable w, i.e. the resultant of G and its partial derivative with respect to w. We write this resultant as $D_w(G) = \text{Res}_w(G, \frac{\partial G}{\partial w})$. Furthermore, the following polynomials are introduced:

$$M(x, z) := \sqrt{D_y(F)} \; , \; R(z) := \begin{cases} 0 & \text{if } \deg_x(M) = 0 \\ D_x(M(x, z)) & \text{otherwise} \end{cases}$$

Result: the main result of [1] is the following (see Theorem 4 and Theorem 13 in [1]):

Theorem 2 *Let S be an algebraic surface implicitly defined by $F \in \mathbb{R}[x, y, z]$, fulfilling the above hypotheses. Then the following statements hold:*

(1) If R is not identically zero, then the set of real roots of R is a critical set of S. If R has no real roots, then the elements of the family show just one topology type.

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(2) *If R is identically zero, then $M = M(z)$, and the set of real roots of M is a critical set of S.*

2.2 Geometrical interpretation of the results for the implicit case.

Let us provide a geometrical interpretation of the process, according to Theorem 2, giving rise to a critical set of S. For this purpose, let $\mathcal{C} = V(F, F_y)$ be the algebraic variety defined by F and its partial derivative F_y (i.e. the intersection of the surfaces defined by F and F_y, respectively). Under the assumptions made on S, one may check that $\gcd(F, F_y) = 1$; so, $\mathcal{C}$ has dimension 1, i.e. it is a space algebraic curve. Taking into account that the normal direction to the surface S at each point $P \in S$ is defined by the gradient vector $\nabla F(P) = (F_x(P), F_y(P), F_z(P))$, one may see that $\mathcal{C}$ consists of the points of S where the normal vector is either parallel to the xz-plane, or identically zero (i.e. singularities of S). Also, let us denote the curve defined by the polynomial $M(x, z)$ on the xz-plane, as $\mathcal{M}$. Now, from a geometric point of view, in order to compute a critical set by means of Theorem 2, one has to perform two different phases:

(1) *Computation of $\mathcal{M}$ (Projection Phase):* from basic properties of resultants (see for example p. 255 in [18]), one may see that the curve $\mathcal{M}$ consists of the projection onto the xz-plane of $\mathcal{C}$, together with the curve defined by the leading coefficient w.r.t. y of F, denoted as $\mathrm{lcoeff}_y(F)$; since by hypothesis $\mathrm{lcoeff}_y(F)$ does not depend on x, this last curve consists of finitely many lines $z - z_0 = 0$ where z_0 is a root of $\mathrm{lcoeff}_y(F)$.

(2) *Computation of the critical set (Analysis of the Projection):* the following real z-values must be computed:
 (i) the z-coordinates of the points of $\mathcal{M}$ with tangent parallel to the x-axis (included the values z_a's so that $z - z_a$ is a component of $\mathcal{M}$).
 (ii) the z-coordinates of the singularities of $\mathcal{M}$
 (iii) the values z_b's so that $z - z_b$ is a horizontal asymptote of $\mathcal{M}$
   In the case $M = M(z)$ this is clear. In other case, one must notice that from well-known properties of resultants (again, p. 255 in [18]), the roots of $\mathrm{Res}_x(M, M_x)$ correspond either to the z-coordinates of the solutions of the polynomial system $\{M(x, z) = 0, M_x(x, z) = 0\}$, or to the roots of $\mathrm{lcoeff}_x(M)$. Then, it suffices to interpret the solutions of the system, and the real roots of $\mathrm{lcoeff}_x(M)$, from a geometric point of view.

In the rest of the paper, we will refer to these z-values as (1)-values, (2)-values and (3)-values, respectively.

In the rest of the paper, let $\mathcal{F}$ be a family of rational algebraic curves, algebraically depending on a parameter λ, defined by the parametric equations

$$\begin{cases} x = u(t, \lambda) \\ y = v(t, \lambda) \end{cases}$$

where u, v are real, rational functions of the variables t, λ in reduced form (i.e. the numerator and denominator of u, v share no common factor), not identically 0. Thus, for almost all real values of λ the above equations define a rational curve of parameter t. In fact, the only exceptions are the values of λ causing that some denominator of u, v vanishes. Now our purpose is to study the topology types arising in $\mathcal{F}$ as λ moves in $\mathbb{R}$. In our approach, the key for computing these topology types is the computation of a finite partition of $\mathbb{R}$ so that for each element of the partition, the topology type of the family stays invariant. Hence, in the sequel we focus on this question. For this purpose, observe that the equations

$$\begin{cases} x = u(t, \lambda) \\ y = v(t, \lambda) \\ z = \lambda \end{cases}$$

define a rational surface S in parametric form, whose level curves are exactly the members of the family $\mathcal{F}$. Hence, our aim is the computation of a critical set of S. For this purpose, we introduce the following notation:

- $\phi_\lambda(t) = (u(t, \lambda), v(t, \lambda))$ is the parametrization of the family $\mathcal{F}$. Hence, the associated surface S is defined by the parametrization $(u(t, \lambda), v(t, \lambda), \lambda)$.
- $F_\lambda(x, y) = F(x, y, \lambda)$ defines the implicit equation of $\mathcal{F}$. Hence, the implicit equation of S is $F(x, y, z) = 0$.

Now in order to make precise the hypotheses that we require on the family $\mathcal{F}$, we need to recall the notion of proper parametrization. One says that a parametrization $\phi(t)$ of a rational curve $\mathcal{H}$ is proper, if there are just finitely many points of $\mathcal{H}$ generated simultaneously by several different values of the parameter t; intuitively speaking, this means that the curve is traced just once as t moves in $\mathbb{R}$. The interested reader may find more information on this topic in Chapter 4.2 of [18]. Then, we consider the following hypotheses:

<u>Hypotheses:</u>

(H_1) The parametrization $\phi_\lambda(t) = (u(t, \lambda), v(t, \lambda))$ is proper for almost all values of λ (i.e. there are just finitely many values of λ such that $\phi_\lambda(t)$ is not proper).

(H_2) $\deg_y(F_\lambda) = \deg(F_\lambda)$, i.e. considering F_λ as a polynomial in the variables x, y, the degree of F_λ w.r.t. the variable y equals the total degree of F_λ; in particular, the leading coefficient of F_λ w.r.t. y does not depend on x, though it may depend on λ.

(H_3) The function $u(t, \lambda)$ depends on the variable t.

In the following, we will refer to these hypotheses as (H_1) and (H_2), respectively. Observe that (H_2) is slightly more restrictive than the hypothesis (c) required in the implicit case. On the other hand, (H_3) guarantees that F_λ (and therefore F) depends on the variable y. Notice that if (H_3) does not hold, the analysis is reduced to the family $x = u(\lambda)$ (consisting just of finitely many lines normal to the x-axis), and the problem is trivial. Finally, since S is a rational surface then it is irreducible; in particular, S cannot represent a plane normal to the z-axis because from the parametrization it is clear that the z-coordinate cannot be constant. Thus, whenever the above hypotheses hold, the surface S described by the considered parametrization satisfies all the hypotheses required in the implicit version of the problem.

The problem of checking the above hypotheses (H_1) and (H_2) is addressed in the next section. Checking (H_3) is trivial; so, in the sequel we assume that (H_3) holds.

3   Checking Hypotheses

In this section we provide an algorithm for checking the hypotheses (H_1) and (H_2) introduced in Subsection 2.3. In addition, the algorithm addressed here provides also a list of finitely many "special" values of the parameter λ, which will be important (see the next section) in order to compute a critical set of the surface S associated with the family.

3.1   Checking hypothesis (H_1)

In this subsection we consider the problem of checking whether (H_1) holds, or not. For this purpose, we start with a technical lemma on the behavior of the gcd of two polynomials of $\mathbb{R}[x, y, \lambda]$, under the specialization of the parameter λ. Hence, let φ_a be the natural homomorphism of $\mathbb{R}[x, y, \lambda]$ into $\mathbb{R}[x, y]$, i.e. for

$a \in \mathbb{R}$,

$$\varphi_a : \mathbb{R}[x, y, \lambda] \rightarrow \mathbb{R}[x, y]$$
$$f(x, y, \lambda) \rightarrow \varphi_a(f) = f(x, y, a)$$

Moreover, if $f, g \in \mathbb{R}[x, y, \lambda]$, we write $f = \bar{f} \cdot \gcd(f, g)$, $g = \bar{g} \cdot \gcd(f, g)$, and we define the following sets:

$$\mathcal{B}_1^{(x,y)}(f, g) = \{a \in \mathbb{R} | \varphi_a(\text{lcoeff}_x(f)) = 0 \text{ or } \varphi_a(\text{lcoeff}_x(g)) = 0\}$$
$$\mathcal{B}_2^{(x,y)}(f, g) = \{a \in \mathbb{R} | \varphi_a(\text{lcoeff}_y(f)) = 0 \text{ or } \varphi_a(\text{lcoeff}_y(g)) = 0\}$$
$$\mathcal{B}_3^{(x,y)}(f, g) = \{a \in \mathbb{R} | \varphi_a\left(\text{Res}_x(\bar{f}, \bar{g})\right)\} = 0$$
$$\mathcal{B}_4^{(x,y)}(f, g) = \{a \in \mathbb{R} | \varphi_a\left(\text{Res}_y(\bar{f}, \bar{g})\right)\} = 0$$
$$\mathcal{B}(f, g) = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$$

Then the following lemma holds.

Lemma 3 *Let $f, g \in \mathbb{R}[x, y, \lambda]$. Then, for all $a \notin \mathcal{B}(f, g)$, it holds that $\gcd(\varphi_a(f), \varphi_a(g)) = \varphi_a(\gcd(f, g))$.*

Proof. Writing $f = \bar{h} \cdot h$, $g = \bar{g} \cdot h$, it holds that

$$\gcd(\varphi_a(f), \varphi_a(g)) = \gcd(\varphi_a(\bar{f}), \varphi_a(\bar{g})) \cdot \varphi_a(h)$$

So, we have to prove that for $a \notin \mathcal{B}(f, g)$, $\gcd(\varphi_a(\bar{f}), \varphi_a(\bar{g})) = 1$. Indeed, if this equality does not hold then either $\text{Res}_x(\varphi_a(\bar{f}), \varphi_a(\bar{g})) = 0$ or $\text{Res}_y(\varphi_a(\bar{f}), \varphi_a(\bar{g})) = 0$. On the other hand, since $a \notin \mathcal{B}(f, g)$ then the resultants behave well under specializations (see Lemma 4.3.1 in [21]), and hence $\text{Res}_x(\varphi_a(\bar{f}), \varphi_a(\bar{g})) = 0$ (resp. $\text{Res}_y(\varphi_a(\bar{f}), \varphi_a(\bar{g})) = 0$) iff $\varphi_a(\text{Res}_x(\bar{f}, \bar{g})) = 0$ (resp. $\varphi_a(\text{Res}_y(\bar{f}, \bar{g})) = 0$). Nevertheless, $\varphi_a(\text{Res}_x(\bar{f}, \bar{g})) = 0$ (resp. $\varphi_a(\text{Res}_y(\bar{f}, \bar{g})) = 0$) cannot happen because $a \notin \mathcal{B}(f, g)$. $\qquad \square$

Now we fix the following notation:

$$\phi_\lambda(t) = (u(t, \lambda), v(t, \lambda)) = \left(\frac{X_{11}(t, \lambda)}{X_{12}(t, \lambda)}, \frac{X_{21}(t, \lambda)}{X_{22}(t, \lambda)}\right),$$

and we introduce the following polynomials:

$$G_1^\phi(t, s, \lambda) = X_{11}(t, \lambda) \cdot X_{12}(s, \lambda) - X_{12}(t, \lambda) \cdot X_{11}(s, \lambda)$$
$$G_2^\phi(t, s, \lambda) = X_{21}(t, \lambda) \cdot X_{22}(s, \lambda) - X_{22}(t, \lambda) \cdot X_{21}(s, \lambda)$$

Moreover, we write $G_1^\phi = \bar{G}_1^\phi \cdot \gcd(G_1^\phi, G_2^\phi)$, $G_2^\phi = \bar{G}_2^\phi \cdot \gcd(G_1^\phi, G_2^\phi)$, and we denote:

$$\mathcal{D}_1 = \mathcal{B}_1^{(t,s)}(G_1^\phi, G_2^\phi), \ \mathcal{D}_2 = \mathcal{B}_2^{(t,s)}(G_1^\phi, G_2^\phi)$$

$$\mathcal{D}_3 = \mathcal{B}_3^{(t,s)}(G_1^\phi, G_2^\phi), \ \mathcal{D}_4 = \mathcal{B}_4^{(t,s)}(G_1^\phi, G_2^\phi)$$

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4,$$

(notice that $\mathcal{D}_1 = \mathcal{D}_2$, $\mathcal{D}_3 = \mathcal{D}_4$ because G_1^ϕ, G_2^ϕ are symmetric w.r.t. t, s). Then, the following theorem holds. Here, we denote the evaluations of G_1^ϕ, G_2^ϕ at $\lambda = a$, as $\varphi_a(G_1^\phi), \varphi_a(G_2^\phi)$, respectively.

Theorem 4 *The parametrization $\phi_\lambda(t)$ is proper for almost all values of λ iff $\deg_t(\gcd(G_1^\phi, G_2^\phi)) = 1$, i.e. iff $\gcd(G_1^\phi, G_2^\phi) = t - s$ (maybe multiplied by some polynomial $\alpha(\lambda) \in \mathbb{R}[\lambda]$). Furthermore, if this condition holds, then the only values of λ where $\phi_\lambda(t)$ may not be proper, are those in $\mathcal{D}$.*

Proof. Let $a \in \mathbb{R}$. By Theorem 4.30 in [18], $\phi_a(t)$ is proper iff

$$\deg_t \left(\gcd \left(\varphi_a(G_1^\phi), \varphi_a(G_2^\phi) \right) \right) = 1.$$

However, by Lemma 3, if $a \notin \mathcal{D}$ then

$$\gcd \left(\varphi_a(G_1^\phi), \varphi_a(G_2^\phi) \right) = \varphi_a \left(\gcd \left(G_1^\phi, G_2^\phi \right) \right)$$

Hence, if we prove that

$$\deg_t \left(\varphi_a \left(\gcd \left(G_1^\phi, G_2^\phi \right) \right) \right) = \deg_t \left(\gcd(G_1^\phi, G_2^\phi) \right)$$

for $a \notin \mathcal{D}$, then we have finished. Indeed, if this equality does not hold, then the leading coefficient w.r.t. t of $\gcd(G_1^\phi, G_2^\phi)$ vanishes at $\lambda = a$. However, in that case the leading coefficients of G_1^ϕ, G_2^ϕ w.r.t. t both vanish at $\lambda = a$, and this cannot happen because $a \notin \mathcal{D}$. Finally, notice that $t - s$ always divides $\gcd(G_1^\phi, G_2^\phi)$; hence, $\deg_t(\gcd(G_1^\phi, G_2^\phi)) = 1$ iff $\gcd(G_1^\phi, G_2^\phi) = t - s$ (perhaps multiplied by some polynomial $\alpha(\lambda)$). $\qquad\square$

Hence, Theorem 4 gives us an algorithm for checking hypothesis (H_1). Moreover, if the condition in Theorem 4 holds one can determine the set $\mathcal{D}$, which contains the finitely many values of the parameter where properness fails. If the condition does not hold, one can compute a reparametrization $\xi_\lambda(t)$ of the family, proper for almost all values of λ, by applying the reparametrization

algorithm in Section 6.1.2, p. 193 of [18]. We give more details in Subsection 3.4.

Remark 1 *Observe that, if they exist, the λ-values making that the denominator of either u or v is identically 0 belong to $\mathcal{D}$.*

3.2   Checking hypothesis (H_2)

Now let us consider hypothesis (H_2). For this purpose, we recall that the *degree of a rational function* (i.e. of a quotient of polynomials) is defined as the maximum of the degrees of the numerator and the denominator; furthermore, the *degree of a rational parametrization* $\phi(t) = (\chi_1(t), \chi_2(t))$ is defined as the maximum of the degrees of $\chi_1(t), \chi_2(t)$ (which are rational functions). We also recall the following result from [18] (see Theorem 4.21 in [18]).

Proposition 5 *If $\phi(t) = (\chi_1(t), \chi_2(t))$, where $\chi_1(t)$ is not identically 0, is a proper rational parametrization of a curve and $f(x, y) = 0$ is its implicit equation, then $\deg_t(\chi_1(t)) = \deg_y(f)$*

Moreover, we also need the following lemma (see for example Section 3 of [10]).

Lemma 6 *Let $f(x, y)$ implicitly define a plane curve $\mathcal{V}$, let $\mu \in \mathbb{R}$, and let $g_\mu(x, y)$ be the implicit equation of the curve that is obtained from $\mathcal{V}$ by applying the linear transformation $\{x = X + \mu Y, y = Y\}$. Then, for almost all values of μ it holds that $\deg_y(g_\mu) = \deg(g_\mu)$ (here, $\deg(g_\mu)$ denotes the total degree of g_μ).*

Now the following result holds. This theorem provides a method for computing the degree of a rational curve just from its parametrization, without making use of its implicit equation (compare also with Theorem 6 and Theorem 7 in [13]).

Theorem 7 *Let $\phi(t) = (\chi_1(t), \chi_2(t))$ be a proper rational parametrization of a curve $\mathcal{V}$, where $\chi_1(t)$ is not identically 0; also let $\chi_\mu(t) = \chi_1(t) - \mu\chi_2(t)$, with μ generic. Then, $\deg(\mathcal{V}) = \deg_t(\chi_\mu(t))$.*

Proof. Let $\mathcal{U}_\mu$ be the curve obtained from $\mathcal{V}$ by applying the linear transformation $\{x = X + \mu Y, y = Y\}$. Then, for all values of μ it holds that $\mathcal{U}_\mu$ is a rational curve properly parametrized by $\tilde{\phi}_\mu(t) = (\chi_\mu(t), \chi_2(t))$. Now let $g_\mu(x, y)$ be the implicit equation of $\mathcal{U}_\mu$. Then by Lemma 6, for a generic μ it holds that $\deg_y(g_\mu) = \deg(g_\mu)$. Now since the degree of a curve is invariant by linear transformations, we have that $\deg(\mathcal{V}) = \deg(g_\mu)$. Finally, since $\chi_1(t), \chi_2(t)$ are not both identically 0, for a generic μ it holds that $\chi_\mu(t)$ is not

identically 0, either; moreover, $\tilde{\phi}_\mu(t)$ is proper for every $\mu \in \mathbb{R}$, and hence by Proposition 5 we have that $\deg_y(g_\mu) = \deg_t(\chi_\mu(t))$. Therefore the statement follows. $\qquad\square$

Theorem 7, together with Proposition 5, provides the following corollary.

Corollary 8 *Let $\phi(t) = (\chi_1(t), \chi_2(t))$, where $\chi_1(t)$ not identically 0, be a proper rational parametrization of a curve $\mathcal{V}$, and let $f(x, y) = 0$ be its implicit equation. Also, let $\chi_\mu(t) = \chi_1(t) - \mu\chi_2(t)$, with μ generic. Then, $\deg_t(\chi_1(t))) = \deg_t(\chi_\mu(t))$ iff $\deg_y(f) = \deg(f)$.*

Then the following characterization of (H_2) can be deduced.

Corollary 9 (H_2) *is fulfilled iff, for a generic μ, it holds that $\deg_t(u(t, \lambda)) = \deg_t(u(t, \lambda) - \mu v(t, \lambda))$. Moreover, the above value provides the degree of F_λ, as a polynomial in the variables x, y.*

Also from [10], one may see that if (H_2) is not satisfied, almost all changes of coordinates of the type $\{x = X + \mu Y, y = Y\}$ set the surface properly. Furthermore, if (H_2) is fulfilled, then the following proposition holds.

Proposition 10 *Assume that (H_2) holds, and let $m = \deg_t(X_{11}(t, \lambda))$, $n = \deg_t(X_{12}(t, \lambda))$, $a(\lambda) = \mathrm{lcoeff}_t(X_{11}(t, \lambda))$, $b(\lambda) = \mathrm{lcoeff}_t(X_{12}(t, \lambda))$. Also, let λ_0 be a real root of the leading coefficient of $F_\lambda(x, y)$ w.r.t. the variable y, so that: (i) $X_{11}(t, \lambda_0)$ does not vanish identically; (ii) $\phi_{\lambda_0}(t)$ is proper. Then, the following statements are true:*

(1) If $m > n$, then $a(\lambda_0) = 0$.
(2) If $m = n$, then $a(\lambda_0) = b(\lambda_0) = 0$.
(3) If $m < n$, then $b(\lambda_0) = 0$.

Proof. Since by hypothesis (H_2) holds, if the leading coefficient of $F_\lambda(x, y)$ vanishes at $\lambda = \lambda_0$ then $\deg_y(F_{\lambda_0}) < \deg_y(F_\lambda)$. Also, because of the required conditions (i) and (ii), Proposition 5 holds and therefore $\deg_y(F_{\lambda_0}) = \deg_t(\chi_1(t, \lambda_0))$. Moreover, from Theorem 7 it holds that $\deg_y(F_\lambda) = \deg_t(\chi_1(t, \lambda))$. So, we deduce that $\deg_t(\chi_1(t, \lambda_0)) < \deg_t(\chi_1(t, \lambda))$. The rest follows from the definition of degree of a rational function. $\qquad\square$

Notice that if $X_{11}(t, \lambda_0) = 0$, then λ_0 is an element of the set $\mathcal{D}$ in Theorem 4. Now Proposition 10 provides the following corollary. Here, we keep the notation used in the above proposition.

Corollary 11 *Assume that (H_1) and (H_2) are fulfilled. Then, the real roots of $\mathrm{lcoeff}_y(F)$ belong to the set consisting of the following elements: (i) the real elements of the set $\mathcal{D}$ in Theorem 4; (ii) the real roots of: (a) $a(\lambda)$, if $m > n$; (b) $b(\lambda)$, if $m < n$; (c) $\gcd(a(\lambda), b(\lambda))$, if $m = n$.*

Given a parametrization $\psi(t)$ of a plane curve $\mathcal{E}$, one says that the parametriza-
tion is normal if it is surjective, i.e. if for all $P_0 \in \mathcal{E}$ there exists $t_0 \in \mathbb{C}$ so that
$\psi(t_0) = P_0$; notice that t_0 may be complex even though P_0 is real. This notion
has been studied in [2], [17], [18]. For our purposes, here we recall the following
result (see Theorem 6.22 in [18]).

Theorem 12 *Let $\phi(t) = \left(\dfrac{X_{11}(t)}{X_{12}(t)}, \dfrac{X_{21}(t)}{X_{22}(t)} \right)$ be a parametrization of a plane
curve $\mathcal{E}$, and let $n = \deg(X_{12})$, $s = \deg(X_{22})$. Also, let $b^\star = \mathrm{coeff}(X_{11}, n)$ (i.e.
the coefficient of degree n in $X_{11}(t)$), $b = \mathrm{coeff}(X_{12}, n)$, $d^\star = \mathrm{coeff}(X_{21}, s)$,
$d = \mathrm{coeff}(X_{22}, s)$. Then,*

(i) If $m > n$ or $r > s$, then $\phi(t)$ is normal.
(ii) If $m \leq n$ and $r \leq s$, then $\phi(t)$ is normal iff

$$\deg_t \left(\gcd \left(b^\star X_{12}(t) - b X_{11}(t), d^\star X_{22}(t) - d X_{21}(t) \right) \right) \geq 1$$

*Moreover, if $\phi(t)$ is not normal, the only point that is not reached by the
parametrization is the so-called "critical point" $\left(\dfrac{b^\star}{b}, \dfrac{d^\star}{d} \right)$, which is a point
of $\mathcal{E}$.*

In our case, we deal with the family

$$\phi_\lambda(t) = (u(t, \lambda), v(t, \lambda)) = \left(\frac{X_{11}(t, \lambda)}{X_{12}(t, \lambda)}, \frac{X_{21}(t, \lambda)}{X_{22}(t, \lambda)} \right).$$

Hence, from Theorem 12 we can derive the following result:

Theorem 13 *Let $m = \deg_t(X_{11})$, $n = \deg_t(X_{12})$, $r = \deg_t(X_{21})$, $s = \deg_t(X_{22})$.
Also, let $a(\lambda) = \mathrm{coeff}(X_{11}(t, \lambda), m)$, $b^\star(\lambda) = \mathrm{coeff}(X_{11}(t, \lambda), n)$, $b(\lambda) = \mathrm{coeff}(X_{12}(t, \lambda), n)$,
$c(\lambda) = \mathrm{coeff}(X_{21}(t, \lambda), r)$, $d^\star(\lambda) = \mathrm{coeff}(X_{21}(t, \lambda), s)$, $d(\lambda) = \mathrm{coeff}(X_{22}(t, \lambda), s)$.
Then, it holds that:*

*(i) If $m > n$ and $r > s$, the values of λ where $\phi_\lambda(t)$ may not be normal satisfy
$\gcd(a(\lambda), c(\lambda)) = 0$.*
*(ii) If $m > n$ and $r \leq s$, the values of λ where $\phi_\lambda(t)$ may not be normal satisfy
$a(\lambda) = 0$.*
*(iii) If $r > s$ and $m \leq s$, the values of λ where $\phi_\lambda(t)$ may not be normal satisfy
$c(\lambda) = 0$.*
(iv) If $m \leq n$, $r \leq s$, and

$$\deg_t \left(\gcd \left(b^\star(\lambda) X_{12}(t) - b(\lambda) X_{11}(t), d^\star(\lambda) X_{22}(t) - d(\lambda) X_{21}(t) \right) \right) \geq 1,$$

the values of λ where $\phi_\lambda(t)$ may not be normal satisfy at least one of the following conditions (here, we denote $\eta(t, \lambda) = \mathrm{lcoeff}_t(b^\star X_{12} - b X_{11})$, $\nu(t, \lambda) = d^\star X_{22} - d X_{21}$, $\tilde{\eta} = \eta / \gcd(\eta, \nu)$, $\tilde{\nu} = \nu / \gcd(\eta, \nu)$): (a) $\mathrm{lcoeff}_t(\eta(t, \lambda)) = 0$; (b) $\mathrm{lcoeff}_t(\nu(t, \lambda)) = 0$; (c) $\mathrm{Res}_t(\tilde{\eta}, \tilde{\nu}) = 0$.

(v) If $n \leq m$, $r \leq s$, the only points of the surface S that may not be reached by the parametrization, are the points of the space curve $\mathcal{C}_{crit}$ parametrized by $\left(\dfrac{b^\star(\lambda)}{b(\lambda)}, \dfrac{d^\star(\lambda)}{d(\lambda)}, \lambda \right)$.

Proof. The statements (i), (ii), (iii) and (v) essentially follow from Theorem 12. Statement (iv) follows from Theorem 12 and Lemma 4.26 in [18]. □

3.4 Summary, and an example

Here we provide the full algorithm for checking the hypotheses (H_1), (H_2), required on the family $\mathcal{F}$, and we illustrate it by means of an example. Besides checking (H_1) and (H_2), the algorithm also provides a list of finitely many 'special" values of the parameter, which will be useful in the next section. If (H_1) does not hold, then one can use the reparametrization algorithm in Section 6.1.2, p. 193 of [18] in the following way (see [18] for further information): (1) let $G^\phi = \gcd(G_1^\phi, G_2^\phi)$; then, choose $(\alpha, \beta) \in \mathbb{R}^2$ so that $G^\phi(\alpha, \beta)$ is not identically 0 (those finitely many λ-values making $G^\phi(\alpha, \beta) = 0$ are incorporated to the list of "special" values); choose also $a, b, c, d \in \mathbb{R}$ so that $ad - bc \neq 0$. (2) Consider the rational function

$$R_\lambda(t) = \frac{a G^\phi(\alpha, t) + b G^\phi(\beta, t)}{c G^\phi(\alpha, t) + d G^\phi(\beta, t)}$$

Then, let $r = \deg(\phi_\lambda) / \deg(R_\lambda(t))$. In general the value of r has to be discussed upon the value of λ, but it will be constant except for finitely many λ-values, which again must be stored in the list of special values. (3) Introduce a generic rational parametrization $\mathcal{Q}(t)$ of degree r (i.e. the generic value of r) with undetermined coefficients. From the equality $\phi_\lambda(t) = \mathcal{Q}(R_\lambda(t))$ derive a linear system of equations in the undetermined coefficients, and by solving it determine $\mathcal{Q}(t)$ (in fact $\mathcal{Q}_\lambda(t)$). Notice that this linear system has λ as a parameter, and therefore, again, certain (special) values of the parameter must also be computed (by discussing the system). In the end, the whole process yields a reparametrization (depending on λ), and a finite list of special values of λ. The correctness of this process follows from Theorem 6.4, p. 191 of [18].

Also, if (H_2) is not satisfied almost all changes of coordinates of the type $\{X = x + \mu y, Y = y\}$ set the surface properly; observe that this kind of transformation leaves the z-coordinate invariant, and so the topology of the

level curves of the surface is not changed.

<u>Algorithm:</u> (Check) Given a uniparametric family $\mathcal{F}$ of rational curves, defined by its parametric equations

$$\phi_\lambda(t) = (u(t,\lambda), v(t,\lambda)) = \left(\frac{X_{11}(t,\lambda)}{X_{12}(t,\lambda)}, \frac{X_{21}(t,\lambda)}{X_{22}(t,\lambda)} \right),$$

where $m = \deg_t(X_{11})$, $n = \deg_t(X_{12})$, $r = \deg_t(X_{21})$, $s = \deg_t(X_{22})$, and $a(\lambda) = \operatorname{coeff}(X_{11}(t,\lambda), m), b^\star(\lambda) = \operatorname{coeff}(X_{11}(t,\lambda), n), b(\lambda) = \operatorname{coeff}(X_{12}(t,\lambda), n),$ $c(\lambda) = \operatorname{coeff}(X_{21}(t,\lambda), r), d^\star(\lambda) = \operatorname{coeff}(X_{21}(t,\lambda), s), d(\lambda) = \operatorname{coeff}(X_{22}(t,\lambda), s),$ and where λ is a real parameter, the algorithm checks hypotheses (H_1) and (H_2); moreover, if the hypotheses are satisfied, the algorithm also computes a list $\mathcal{S}pec$ of finitely many "special" values of λ.

1. (Check hypothesis (H_1))
   1.1 Compute the polynomials G_1^ϕ, G_2^ϕ in Section 3.1.
   1.2 If $\deg_t(\gcd(G_1^\phi, G_2^\phi)) = 1$, then return (H_1) holds, otherwise return (H_1) does not hold.
   1.3 If (H_1) does not hold, reparametrize the family. Let $\mathcal{S}pec_0$ be the list of special values computed in the process (if (H_1) holds, $\mathcal{S}pec_0 = \emptyset$).
   1.4 Compute the set $\mathcal{D}$ in Theorem 4, and let $\mathcal{S}pec_1 := \mathcal{D}$.
2. (Check hypothesis (H_2))
   2.1 Compute $\deg_t(u(t,\lambda)), \deg_t(u(t,\lambda) - \mu v(t,\lambda))$. If both are equal, then return (H_2) holds, otherwise return (H_2) does not hold.
   2.2 If (H_2) does not hold, apply a change of coordinates $\{X = x + \mu y, Y = y\}$, and go back to 1.3.
   2.3 If (H_2) holds, then compute the real roots of: (a) $a(\lambda)$, if $m > n$; (b) $b(\lambda)$, if $m < n$; (c) $\gcd(a(\lambda), b(\lambda))$, if $m = n$. Let $\mathcal{S}pec_2$ be the set consisting of these values.
3. (Normality)
   3.1 If $m > n$ and $r > s$, let $\mathcal{S}pec_3$ be the set of real roots of $\gcd(a(\lambda), c(\lambda))$.
   3.2 If $m > n$ and $r \le s$, let $\mathcal{S}pec_3$ be the set of real roots of $a(\lambda)$.
   3.3 If $r > s$ and $m \le n$, let $\mathcal{S}pec_3$ be the set of real roots of $c(\lambda)$.
   3.4 If If $m \le n$, $r \le s$, then compute

   $$\delta = \deg_t \left(\gcd \left(b^\star(\lambda) X_{12}(t) - b(\lambda) X_{11}(t), d^\star(\lambda) X_{22}(t) - d(\lambda) X_{21}(t) \right) \right)$$

   3.4.1 If $\delta \ge 1$, then let $\eta(t,\lambda) := \operatorname{lcoeff}_t(b^\star X_{12} - b X_{11}), \nu(t,\lambda) := d^\star X_{22} - d X_{21}, \tilde\eta := \eta/\gcd(\eta,\nu), \tilde\nu := \nu/\gcd(\eta,\nu),$ and let $\mathcal{S}pec_3$ be the set of real roots of $\operatorname{lcoeff}_t(\eta(t,\lambda)), \operatorname{lcoeff}_t(\nu(t,\lambda)), \operatorname{Res}_t(\tilde\eta, \tilde\nu) = 0$.
   3.4.2 If $\delta < 1$ then determine the parametrization

   $$\left(\frac{b^\star(\lambda)}{b(\lambda)}, \frac{d^\star(\lambda)}{d(\lambda)}, \lambda \right)$$

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of the space curve $\mathcal{C}_{crit}$. Let $\mathcal{S}pec_3 := \emptyset$

4. **Special values:** $\mathcal{S}pec := \mathcal{S}pec_0 \cup \mathcal{S}pec_1 \cup \mathcal{S}pec_2 \cup \mathcal{S}pec_3$

The above algorithm is illustrated by the next example. Here, we consider the parametric equation of the offset family to a cardioid. By using the results in [3], one may check that this offset family is rational; moreover, the parametrization used in the example is taken from [16].

Example 1 *Let $\mathcal{O}_d(\mathcal{C})$ be the offset family to the cardioid $(x^2 + 4y + y^2)^2 - 16(x^2+y^2) = 0$. Then, a parametrization of this family is $\phi_d(t) = (u(t,d), v(t,d))$, where:*

$$u(t,d) = \frac{3456t^5 - 31104t^3 + dt^8 - 126dt^6 + 10206dt^2 - 6561d}{486t^4 + 36t^6 + 2916t^2 + t^8 + 6561}$$

$$v(t,d) = \frac{(-18)t(864t^3 - 16t^5 - 1296t + dt^6 - 21dt^4 - 189dt^2 + 729d)}{486t^4 + 36t^6 + 2916t^2 + t^8 + 6561}$$

Here, the parameter d denotes the offsetting distance. So, let us apply the algorithm Check. *In step (1), we analyze (H_1). For this purpose, we compute G_1^ϕ, G_2^ϕ (step 1.1):*

$G_1^\phi = (3456t^5 - 31104t^3 + dt^8 - 126dt^6 + 10206dt^2 - 6561d)(486s^4 + 36s^6 + 2916s^2 + s^8 + 6561) - (3456s^5 - 31104s^3 + ds^8 - 126ds^6 + 10206ds^2 - 6561d)(486t^4 + 36t^6 + 2916t^2 + t^8 + 6561)$

$G_2^\phi = -18t(864t^3 - 16t^5 - 1296t + dt^6 - 21dt^4 - 189dt^2 + 729d)(486s^4 + 36s^6 + 2916s^2 + s^8 + 6561) + 18s(864s^3 - 16s^5 - 1296s + ds^6 - 21ds^4 - 189ds^2 + 729d)(486t^4 + 36t^6 + 2916t^2 + t^8 + 6561)$

Then, we check that $\gcd(G_1^\phi, G_2^\phi) = t - s$, and therefore that $\deg_t(\gcd(G_1^\phi, G_2^\phi)) = 1$ (step (1.2)); hence, (H_1) holds, $\mathcal{S}pec_0 := \emptyset$, and we go to step (1.4). Here, we compute the set $\mathcal{D}$ containing the d-values where $\phi_d(t)$ may not be proper; for this purpose, we compute the real roots of: (i) $\mathrm{Content}_s(\mathrm{lcoeff}(G_1^\phi, t))$; (ii) $\mathrm{Content}_s(\mathrm{lcoeff}(G_2^\phi, t))$; (iii) $\mathrm{Content}_t\left(\mathrm{Res}_t(\bar{G}_1^\phi, \bar{G}_2^\phi)\right)$. In this case, we get that $\mathrm{Content}_s(\mathrm{lcoeff}(G_1^\phi, t)) = 54$, $\mathrm{Content}_s(\mathrm{lcoeff}(G_2^\phi, t)) = 18$, and $\mathrm{Res}_t(\bar{G}_1^\phi, \bar{G}_2^\phi) = C \cdot d(t^2 + 9)^{44}(729d - 1053dt^2 - 117dt^4 + dt^6 + 3456t^3)(-32t + dt^2 + 9d)^2$ (with $C \in \mathbb{N}$). Hence, it holds that $\mathcal{D} = \{0\}$ and therefore

$$\mathcal{S}pec_1 := \{0\}.$$

Now in step (2), we check (H_2). For this purpose, in step (2.1) we compute $\deg_t(u(t,d)) = 8$ and $\deg_t(u(t,d) - \mu v(t,d)) = 8$; since both of them coincide, then (H_2) holds, and we go to step (2.3). Since $m = n = 8$, we compute

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$a(d) = d$, $b(d) = 1$, *and we determine* $\gcd(a(d), b(d))$, *which is 1; hence,*

$$\mathcal{S}pec_2 := \emptyset.$$

Finally, in step (3) we consider normality questions. We have that $r = 7, s = 8$ *and therefore* $r < s$. *Since* $m = n$, *we go to (3.4) and we compute*

$$\eta(t, d) = d(486t^4 + 36t^6 + 2916t^2 + t^8 + 6561) - 3456t^5 + 31104t^3 - dt^8 + 126dt^6$$
$$-10206dt^2 + 6561d$$
$$\nu(t, d) = 18dt(864t^3 - 16t^5 - 1296t + dt^6 - 21dt^4 - 189dt^2 + 729d)$$
$$\delta = \deg_t(\gcd(\eta, \nu))$$

We get $\delta = 0$; *hence, we go to step (3.4.2) and we obtain a space curve of possibly non-reached points, namely*

$$\mathcal{C}_{crit} = (d, 0, d)$$

Finally, we get that $\mathcal{S}pec := \{0\}$.

4 The algorithm for the parametric case.

Based on the observations made in Subsection 2.2, in this section we present an algorithm for computing a critical set of S, under the assumption that the hypotheses requested in Subsection 2.3 are satisfied; the algorithm does not compute or make use of the implicit equation of F. More precisely, in Subsection 2.2 it is observed that one can compute a critical set of S by determining certain z-values referred to as (1)-values, (2)-values and (3)-values, respectively. These values have different geometric meanings, and are related to notable points and lines of the curve $\mathcal{M}$ defined in the xz-plane by the polynomial $M(x, z) = \sqrt{D_y(F)}$, where F is the implicit equation of S. On the other hand, also in Subsection 2.2 it is shown that $\mathcal{M}$ can be written as the union of the projection onto the xz-plane of the space curve $\mathcal{C} = V(F, F_y)$, and the curve (in the xz-plane) defined by $\operatorname{lcoeff}_y(F)$. Now, the real roots of $\operatorname{lcoeff}_y(F)$ (or, more precisely, a finite set containing them) can be determined by means of Corollary 11, without explicitly computing F. Hence, in this section we focus on the computation of the remaining (1), (2) and (3)-values, which are related to $\mathcal{C}$; for this purpose, and since we do not want to compute or make use of F, the idea is to work with a "parametric description" of $\mathcal{C}$.

In order to provide equations for $\mathcal{C}$, one may see (recall the first paragraph in Subsection 2.2) that this curve consists of the following points:

- points of S, reached by the parametrization, such that the normal vector to S is either $\vec{0}$ (in which case the point is a singularity of S), or parallel to the xz plane.
- self-intersections of S, reached by the parametrization.
- points of S with some of the above geometric properties, but not reached by the parametrization.

In the sequel we will refer to these sets as **first**, **second** and **third** sets, respectively. Notice that some of these sets may be empty, and that they are not necessarily disjoint. So, we start with the first one. By performing easy computations with the parametrization $(u(t, \lambda), v(t, \lambda), \lambda)$ of S, one may see that the expression of the normal vector to S is:

$$\vec{N} = v_t \vec{i} - u_t \vec{j} + (u_t v_\lambda - u_\lambda v_t)\vec{k}$$

Hence, we define $\mathcal{C}_1$ as the following subset of points $(x, y, z) \in \mathbb{C}^3$:

$$\mathcal{C}_1 \equiv \begin{cases} x = u(t, \lambda) \\ y = v(t, \lambda) \\ z = \lambda \\ h(t, \lambda) = 0 \\ X_{12}(t, z) \cdot X_{22}(t, z) \neq 0 \end{cases},$$

where $h(t, \lambda)$ is the square-free part of the numerator of $u_t(t, \lambda)$. One may see that $\mathcal{C}_1$ contains the first set. Then let us consider the second set (i.e. self-intersections of S). For this purpose, we impose

$$u(t, \lambda) = u(s, \lambda), v(t, \lambda) = v(s, \lambda), \text{ with } t \neq s$$

Hence, we find again the polynomials G_1^ϕ, G_2^ϕ introduced in Subsection 3.1. We recall from Subsection 3.1 the notation $\bar{G}_1^\phi, \bar{G}_2^\phi$ for the polynomials obtained by removing the common factor $t - s$ in G_1^ϕ, G_2^ϕ. Furthermore, we write $j(t, \lambda) = \sqrt{\text{Res}_s(\bar{G}_1^\phi, \bar{G}_2^\phi)}$. Since by hypothesis the parametrization $\phi_\lambda(t) = (u(t, \lambda), v(t, \lambda), \lambda)$ is proper for almost all values of λ, then $j(t, \lambda)$ cannot be identically 0 (see Theorem 4.30 in [18]). Therefore, we define $\mathcal{C}_2$ as the following

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subset of points $(x, y, z) \in \mathbb{C}^3$:

$$
\mathcal{C}_2 \equiv \begin{cases}
x = u(t, \lambda) \\
y = v(t, \lambda) \\
z = \lambda \\
j(t, \lambda) = 0 \\
X_{12}(t, z) \cdot X_{22}(t, z) \neq 0
\end{cases}
$$

One may see that this set contains the self-intersections of S reached by the parametrization. Finally, the third set of points has been studied in Subsection 3.3. In this sense, by Theorem 13 we have basically two possibilities: in the first case (see statements (i), (ii), (iii), (iv) in Theorem 13), there exists a finite set $\mathcal{N}$ containing the λ-values so that $\phi_\lambda(t)$ may not be normal; in that situation, we just add $\mathcal{N}$ to the rest of elements of the critical set, and therefore no special difficulty arises. In the second case (statement (v) of Theorem 13), it may happen that there are infinitely many λ values so that $\phi_\lambda(t)$ is not normal; thus, the non-reached points of S are among the points of the space curve $\mathcal{C}_{crit}$ parametrized by $\left(\dfrac{b^\star(\lambda)}{b(\lambda)}, \dfrac{d^\star(\lambda)}{d(\lambda)}, \lambda \right)$. Hence, in the sequel we will assume that we are in this more complicated situation.

4.1 Computation of a critical set

Since the roots of $\mathrm{lcoeff}_y(F)$ can de determined from Corollary 11, we focus on the remaining elements of the critical set. For this purpose, we need to analyze the projection of $\mathcal{C}$ onto the xz-plane. From the above reasonings, we can write

$$
\mathcal{C} \subset \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_{crit}
$$

So, $\pi_{xz}(\mathcal{C}) \subset \pi_{xz}(\mathcal{C}_1) \cup \pi_{xz}(\mathcal{C}_2) \cup \pi_{xz}(\mathcal{C}_{crit})$. Now we have that

$$
\pi_{xz}(\mathcal{C}_1) \equiv \begin{cases}
x = u(t, z) \\
h(t, z) = 0 \\
X_{12}(t, z) \cdot X_{22}(t, z) \neq 0
\end{cases}
$$

Let $f(x, t, z)$ be the numerator of $x - u(t, z)$. Also, we write $h = \tilde{h} \cdot \gcd(h, X_{12} \cdot X_{22})$. Then, we denote the curve defined by $m^{(1)}(x, z) = \sqrt{\mathrm{Res}_t(f, \tilde{h})}$ as $\mathcal{M}_1$;

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notice that $m^{(1)}(x, z)$ cannot be identically 0 because by hypothesis the function $u(t, \lambda)$ is given in reduced form (i.e. its numerator and denominator share no common factor). Furthermore, one may see that $\pi_{xz}(\mathcal{C}_1)$ is included in $\mathcal{M}_1$. Similarly,

$$
\pi_{xz}(\mathcal{C}_2) \equiv \begin{cases} x = u(t, z) \\ j(t, z) = 0 \\ X_{12}(t, z) \cdot X_{22}(t, z) \neq 0 \end{cases}
$$

In this case, we write $j = \tilde{j} \cdot \gcd(h, X_{12} \cdot X_{22})$, we denote $m^{(2)}(x, z) = \sqrt{\text{Res}_t(f, \tilde{j})}$ (which cannot be identically 0 for the same reason than $m^{(1)}(x, z)$), and we represent the curve defined by $m^{(2)}$ as $\mathcal{M}_2$. One may observe that $\pi_{xz}(\mathcal{C}_2)$ is included in $\mathcal{M}_2$. Finally, $\pi_{xz}(\mathcal{C}_{crit})$ is the parametric curve defined by $\left(\dfrac{b^\star(\lambda)}{b(\lambda)}, \lambda \right)$.

Now, the z-coordinates of the singularities and of the points with tangent parallel to the x-axis of $\mathcal{M}_1$ (resp. $\mathcal{M}_2$), and also the values z_a so that $z - z_a$ is a horizontal asymptote of $\mathcal{M}_1$ (resp. $\mathcal{M}_2$), correspond to real roots of $\text{Res}_x(m^{(1)}, m_x^{(1)})$ (resp. $\text{Res}_x(m^{(2)}, m_x^{(2)})$). On the other hand, since $\pi_{xz}(\mathcal{C}_{crit})$ is parametrized by $\left(\dfrac{b^\star(\lambda)}{b(\lambda)}, \lambda \right)$, it has no point with tangent parallel to the x-axis, no singularity, and no component parallel to the x-axis. However, it may have horizontal asymptotes, corresponding to the roots of $b(\lambda)$.

Finally, in order to compute a critical set we also need to determine the intersections between $\mathcal{M}_1$ and $\mathcal{M}_2$, $\mathcal{M}_1$ and $\pi_{xz}(\mathcal{C}_{crit})$, $\mathcal{M}_2$ and $\pi_{xz}(\mathcal{C}_{crit})$, respectively. In the first case, let

$$
\overline{m^{(1)}} = m^{(1)} / \gcd(m^{(1)}, m^{(2)}), \overline{m^{(2)}} = m^{(2)} / \gcd(m^{(1)}, m^{(2)});
$$

then, the z-coordinates of the intersections between $\mathcal{M}_1$ and $\mathcal{M}_2$ are contained in the set of roots of $\text{Res}_x(\overline{m^{(1)}}, \overline{m^{(2)}})$. In the second (resp. the third) case, we simply compute the roots of the numerator $M^{(1)}(z)$ (resp. $M^{(2)}(z)$) where $M^{(1)}(z)$ (resp. $M^{(2)}(z)$) is defined as the result of substituting $x := \dfrac{b^\star(\lambda)}{b(\lambda)}$, $z := \lambda$ in $m^{(1)}(x, z)$ (resp. $m^{(2)}(x, z)$), whenever this substitution does not yield 0, and is defined as 1 in other case.

So, we can derive the following algorithm for computing a critical set of $\mathcal{F}$. Here, we will use the subset $\mathcal{S}pec$ computed by the algorithm Check, that we developed in Section 3.

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<u>Algorithm</u>: (`Critical`) Given a uniparametric family $\mathcal{F}$ of rational curves depending on a parameter λ, defined by its parametric equations

$$\phi_\lambda(t) = (u(t,\lambda), v(t,\lambda)) = \left(\frac{X_{11}(t,\lambda)}{X_{12}(t,\lambda)}, \frac{X_{21}(t,\lambda)}{X_{22}(t,\lambda)}\right),$$

fulfilling hypotheses $(H_1), (H_2), (H_3)$, where $m = \deg_t(X_{11})$, $n = \deg_t(X_{12})$, $r = \deg_t(X_{21})$, $s = \deg_t(X_{22})$, and $b^\star(\lambda) = \mathrm{coeff}(X_{11}(t,\lambda), n)$, $b(\lambda) = \mathrm{coeff}(X_{12}(t,\lambda), n)$, $d^\star(\lambda) = \mathrm{coeff}(X_{21}(t,\lambda), s)$, $d(\lambda) = \mathrm{coeff}(X_{22}(t,\lambda), s)$, the algorithm `computes` a critical set $\mathcal{A}$ of the family.

(1) Compute the set $\mathcal{A}_1$ consisting of the real roots of the following polynomials:
   · $\mathrm{Res}_x(m^{(1)}, m_x^{(1)})$
   · $\mathrm{Res}_x(m^{(2)}, m_x^{(2)})$.
   · $\mathrm{Res}_x(m^{(1)}, m^{(2)})$.
(2) If $m > n$, or $r > s$, or $m \le n, r \le s$ but

$$\deg_t(\gcd(b^\star(\lambda)X_{12}(t) - b(\lambda)X_{11}(t), c(\lambda)X_{22}(t) - d(\lambda)X_{21}(t))) \ge 1,$$

then $\mathcal{A}_2 = \emptyset$. Otherwise, let $\mathcal{A}_2$ be the set consisting of the real roots of the following polynomials:
   · $b(\lambda)$.
   · $M^{(1)}(z)$
   · $M^{(2)}(z)$
(3) Let $\mathcal{A} = \mathcal{S}pec \cup \mathcal{A}_1 \cup \mathcal{A}_2$. `Return` $\mathcal{A}$.

4.2 Correctness of the Algorithm

The aim of this subsection is to prove that the algorithm `Critical`, provided in the above subsection, is correct, i.e. that the set $\mathcal{A}$ determined by the algorithm is a critical set of S. The necessity of this proof is due to the fact that in `Critical` we are working not really with $\pi_{xz}(\mathcal{C})$, but with the curve $\mathcal{M}^\star = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \pi_{xz}(\mathcal{C}_{crit})$, which in general may be bigger than $\pi_{xz}(\mathcal{C})$. Hence, we want to ensure that no critical value (i.e. no λ-value where the topology of the family changes) has been missed when passing from $\pi_{xz}(\mathcal{C})$ to $\mathcal{M}^\star$. This is done in the following theorem.

Theorem 14 *The algorithm* `Critical` *is correct.*

Proof. Since $\pi_{xz}(\mathcal{C}) \subset \mathcal{M}^\star$, from Chapter 3, Section 2.5, Exercise 5 of [20] it follows that the singularities of $\pi_{xz}(\mathcal{C})$ are also singularities of $\mathcal{M}^\star$, and hence their z-coordinates are found by the algorithm. Now, let $Q \in \pi_{xz}(\mathcal{C})$ so that the tangent line ℓ to $\pi_{xz}(\mathcal{C})$ at Q is parallel to the x-axis. If Q is a regular point of $\mathcal{M}^\star$, since $\pi_{xz}(\mathcal{C}) \subset \mathcal{M}^\star$ we have that ℓ is also the tangent line to $\mathcal{M}^\star$ at

Q; therefore Q is a point of $\mathcal{M}^\star$ with tangent parallel to the x-axis, and hence its z-coordinate is found by the algorithm. Otherwise, Q is a singular point of $\mathcal{M}^\star$ and therefore its z-coordinate is also determined by the algorithm. On the other hand, asymptotic branches of $\pi_{xz}(\mathcal{C})$ and components lying in planes normal to the z-axis, keep their nature when passing to $\mathcal{M}^\star$. Finally, since the real roots of $\mathrm{lcoeff}_y(F)$ are included in $\mathcal{S}pec$, we conclude that all the (1), (2) and (3)-values are computed by `Critical`, and the result follows. $\qquad\square$

Remark 2 *Notice that we are stating that the set computed by the algorithm* `Critical` *is a critical set, i.e. that it contains all the λ-values where the topology type of the family may change. But we are not stating that the set that we are computing is optimal. So, the output of the algorithm may include additional z-values where the topology of the family does not change. This drawback was already present in the algorithm of [1], which was not optimal, either. However, our algorithm may yield critical sets which are bigger than those in [1]. Since affine transformations preserve the topology of curves, one can recognize some superfluous values in a computed critical set in the following way: (1) apply a random affine transformation $\{X = x + \mu y, Y = y\}$ to the family; (2) compute a critical set of the new family; (3) discard those values of the original critical set that do not belong to the new critical set.*

4.3 Experimentation and Results

The algorithm `Critical` has been implemented in Maple, and tested with several examples. In this subsection, first we continue the analysis of the topology types in the offset family to the cardioid, started in Example 1. Then we provide a simple example illustrating the non-optimality of the algorithm, as mentioned in Remark 2. Finally we present a table comparing timings between the algorithm for the implicit case, deducible from [1], and our algorithm.

Example 1 (continued): *By applying the algorithm* `Critical`, *one determines the following critical set:*

$$\mathcal{A} = \{-16/3, -\alpha, -3\sqrt{3}, -8\sqrt{3}/3, -3\sqrt{3}/2, 0, 3\sqrt{3}/2, 8\sqrt{3}/3, 3\sqrt{3}, \alpha, 16/3\}$$

(which coincides with the output of the implicit algorithm), where α is the real root of $729\lambda^5 - 1215\lambda^4 + 702\lambda^3 - 18\lambda^2 + 13\lambda - 27$. The total amount of time required for this computation was 1.5 seconds (the cost of checking the hypotheses is included). From this critical set, one may deduce that there are at most 19 different topology types in the family. However, by applying a random linear transformation as suggested in Remark 2, one can compute a reduced

critical set, namely

$$\{-16/3, -3\sqrt{3}, 0, 3\sqrt{3}, 16/3\}$$

In this case, because of the properties of offset curves, one has that for $d = 0$ one gets the original curve, i.e. a cardioid, and that for d_0 and $-d_0$, the shape is the same. So, in our analysis we have just considered positive values of d. In Figure 2 one may find the different shapes arising in the family, and the intervals corresponding to each of them. In the first row (at the top of the figure), we display the pictures (i), (ii), (iii) corresponding to the distances $d = 1$, $d = 3$, $d = 22/5$, respectively, all of them belonging to the interval $(0, 3\sqrt{3}/2)$ and therefore sharing the same topology type (we have plotted the three pictures so as to clearly see the evolution of the family, as d is increased). In the second row, from left to right we have the picture (iv) corresponding to $d = 3\sqrt{3}$, the picture (v) corresponding to a distance $d \in (3\sqrt{3}, 16/3)$, and (vi), that corresponds to $d = 16/3$. In the third row, the shape (vii) corresponding to $d > 16/3$ is shown. Also, in each figure we have included the plotting of the original cardioid. One may see that the offsets exhibit two cusps for $d < 3\sqrt{3}$, and a loop for $d \geq 16/3$. However, the topologies of (iv) and (v) are not completely clear, since the picture does not show well enough the behavior next to the singularity with negative y-coordinate. If one enlarges the part of the curve next to this singularity, one obtains the pictures in Figure 3. Here we have plotted a detail of (iv) (left), of (v) (middle), and of (vi) (right). So, in (iv) there is a non-ordinary singularity; in (v) there is not one, but two singularities, corresponding to two self-intersections of the curve, giving rise to two different loops; in (vi), the topology changes so that the curve has only one loop (the origin, in this case, is a singular point).

Example 2: *Consider the family $\phi_d(t) = (u(t, d), v(t, d))$, where:*

$$u(t, d) = -25 + 11t^2 - 29t + d(57 - 95t^2 - 22t)$$
$$v(t, d) = 49 + 18t^2 + 51t + d(70 + 34t^2 - 64t)$$

The implicit equation of the family is $1136239 - 393995d - 19629x - 53165y + 202885dx + 130530992d^3 - 1200232d^2x + 374269dy - 2090dy^2 + 121y^2 + 59360320d^4 + 324x^2 - 396yx - 33513124d^2 + 1156d^2x^2 + 1224dx^2 - 688992d^2y - 1781936yd^3 + 9025d^2y^2 + 2672dxy + 6460yd^2x - 146992xd^3 = 0.$

Now by applying the algorithm in [1] we get that $\left\{\dfrac{11}{25}\right\}$ is a critical set. However, Critical yields $\left\{\dfrac{11}{25}, \dfrac{-9}{17}\right\}$ (which in this case coincides with $Spec_1$, see Subsection 3.4); so, the second element of this last set is clearly superfluous. In fact, when plotting curves corresponding to $d \in (-\infty, 11/25)$, $d = 11/25$
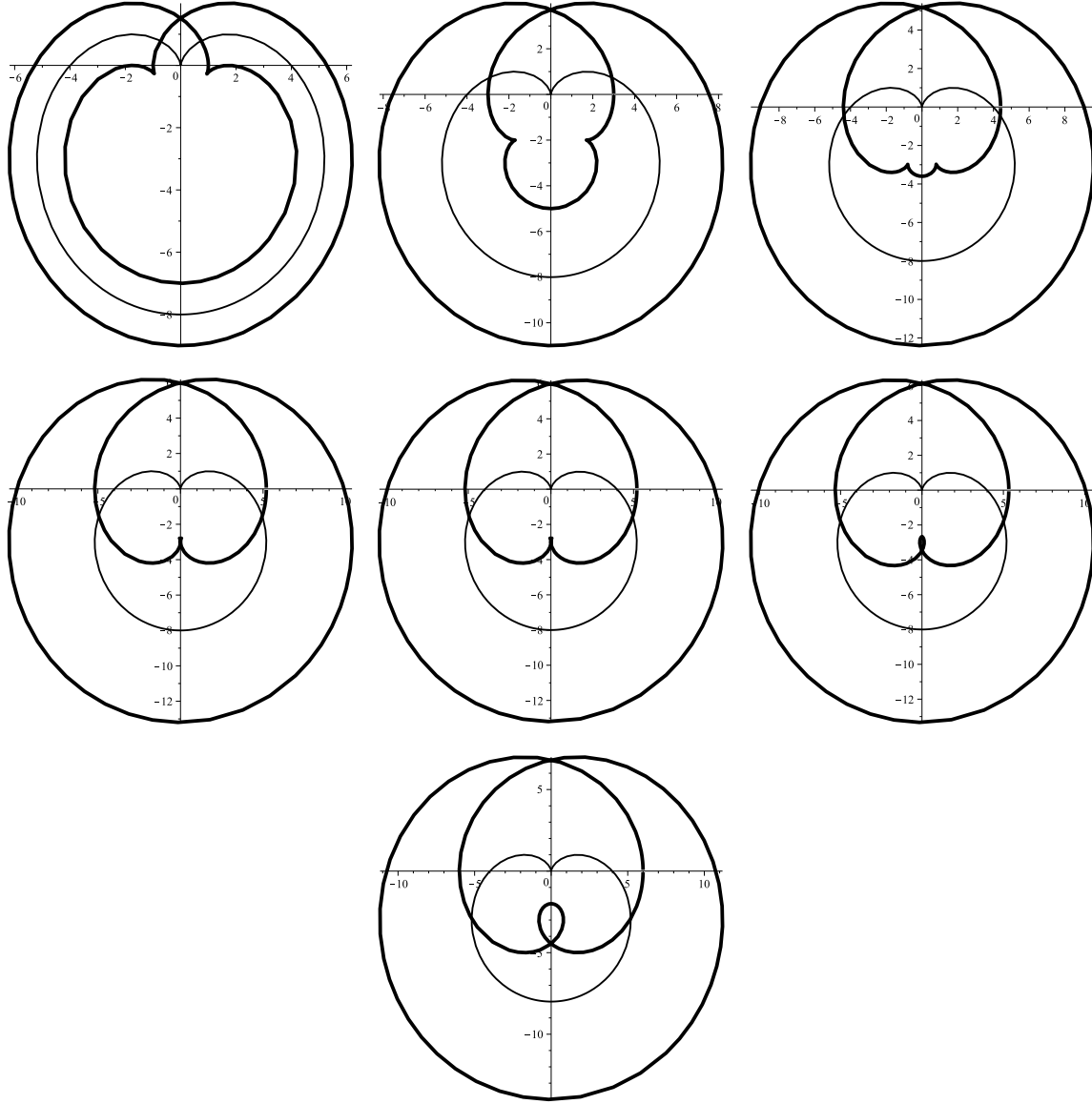
Fig. 2. Offsets to the cardioid

and $d \in (11/25, \infty)$, one gets a parabola in all the cases; hence, the topology type does not change for $d \in \mathbb{R}$, i.e. even the value $11/25$ provided by the implicit algorithm, is superfluous. The total amount of time required for the whole computation is 0.094 seconds.

Comparison Table. The following table shows a comparison between the algorithm derived from the results in [1], and our algorithm, both of them implemented in Maple. For each family, in this table we include whether the parametrization is rational or polynomial (Type: R, rational, or P, polynomial), and we provide the following data: the degree of the implicit equation of the associated surface ($\deg(f)$), the total degree of the parametrization ($\deg_t(\phi)$) i.e. the greatest power of the parameter t arising in the numerators
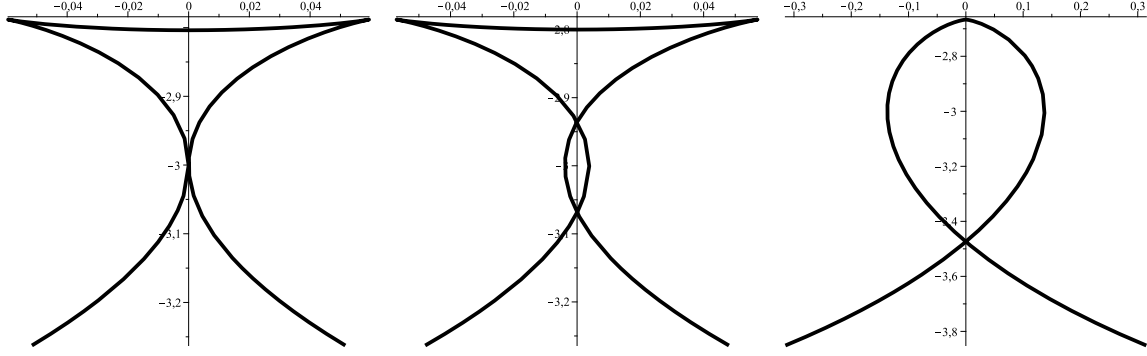
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Fig. 3. Details of Some Offsets to the cardioid

and denominators of the coordinates, the highest power $(\deg_\lambda(\phi))$ of λ arising in the numerators and denominators of the coordinates, the timing for the algorithm in [1] (Imp., in seconds), the timing for our algorithm (Crit., in seconds), the size of the critical set determined by the parametric algorithm (Size), and the difference (Dif.) between the sizes of the critical sets provided by both algorithms (i.e. the size of the critical set provided by `Critical` minus the size of the critical set provided by the algorithm in [1]). The symbol * in the column of Imp. means that the algorithm has been unable to provide an answer, or that the computation time exceeded reasonable bounds. In Appendix I one may find the expressions of all the parametrizations used here. It is worth mentioning that except for the families numbers 3 and 7, the rest of the examples have been randomly generated. Also, the timings given include the cost of checking the hypotheses.

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Family	Type	$\deg(f)$	$\deg_t(\phi)$	$\deg_\lambda(\phi)$	Imp.	Crit.	Size	Dif.
1	P.	8	4	1	35.347	1.045	14	1
2	P.	10	5	1	160.664	110.076	36	1
3	R.	22	7	4	*	252.390	70	*
4	R.	9	6	1	*	35.990	33	*
5	P.	10	5	1	158.140	108.389	36	1
6	P.	28	5	4	39.983	14.945	21	0
7	R.	16	5	2	*	31.543	33	*
8	R.	9	3	2	9.157	0.344	12	3
9	R.	24	6	3	*	74.770	78	*
10	R.	6	2	2	0.109	0.093	7	2
11	R.	12	3	3	19.580	0.389	25	0
12	R.	18	3	3	66.253	1.325	30	9

4.4   Improvements in the computation.

The (1)-values, (3)-values and (2)-values not corresponding to self-intersections of $\mathcal{M}$ can be more efficiently computed by taking advantage of certain geometric properties of $\mathcal{C}$. In fact, one can determine these values by solving polynomial systems in two variables; so, we can avoid one resultant, and identify quite fast some values as potentially critical. This is based on two classical results. The first one follows essentially from Proposition 3 of [7].

Proposition 15 *Let $Q = (x_q, z_q)$ be a singularity of $\mathcal{M}$, which is not a self-intersection of $\mathcal{M}$, and such that z_q is not a root of $\mathrm{lcoeff}_y(F)$. Then, one of these two possibilities occur: (i) Q is the projection of a singularity of $\mathcal{C}$; (ii) there exists a point of $\mathcal{C}$, projecting onto Q, so that the tangent to $\mathcal{C}$ at this point is normal to the xz-plane.*

The second result relates the non-singular points of $\mathcal{M}$ with tangent parallel to the z-axis, to certain notable points of $\mathcal{C}$. It can be easily proven by reasoning with places.

Proposition 16 *Let $Q \in \mathcal{M}$ be a non-singular point of $\mathcal{M}$ with tangent parallel to the x-axis. Then, there exists some point $Q' \in \mathcal{C}$, projecting onto Q, so that the tangent to $\mathcal{C}$ at Q' is parallel to the xz-plane.*

So, let us consider first (1)-values and (2)-values. Those of these values not corresponding to: (i) self-intersections of $\mathcal{M}$, (ii) real roots of $\mathrm{lcoeff}_y(F)$, (iii) points of $\pi_{crit}(\mathcal{C})$, are real z_0-values fulfilling that there exists $(x_0, z_0) \in \mathcal{M}_1 \cup \mathcal{M}_2$. Now $\mathcal{M}_1$ can be seen as the union of the following two curves: (1) the projection onto the xz-plane of the space curve $\tilde{\mathcal{C}}_1$ defined by $f(x, t, z) = 0$, $h(t, z) = 0$ in the Euclidean space with coordinates $\{x, t, z\}$, which we denote as $\pi_{xz}(\tilde{\mathcal{C}}_1)$; (2) the curve defined in the xz-plane by $\gcd(\mathrm{lcoeff}_t(h), \mathrm{lcoeff}_t(f))$. The equation of $\pi_{xz}(\tilde{\mathcal{C}}_1)$ is clearly $h(t, z) = 0$. Thus, by Proposition 15 and Proposition 16, and using elementary properties of the resultant, one gets that the considered values belonging to $\mathcal{M}_1$ also satisfy $h_t(t, z) = 0$, and hence they are contained in the set of real roots of $\mathrm{Res}_t(h, h_t)$; one may observe that this set contains also the real roots of $\gcd(\mathrm{lcoeff}_t(h), \mathrm{lcoeff}_t(f))$. Arguing in a similar way for $\mathcal{M}_2$ we would reach the condition $\mathrm{Res}_t(j, j_t) = 0$. Moreover, the (3)-values can be related with the asymptotes of the curves (in the tz-plane) $h(t, z) = 0$ and $j(t, z) = 0$. So, the following theorem holds.

Theorem 17 *The (1)-values, (2)-values not corresponding to self-intersections of $\mathcal{M}$, and (3)-values not corresponding to asymptotes of $\pi_{xz}(\mathcal{C}_{crit})$, are among the finitely many real roots of $\mathrm{Res}_t(h, h_t)$, $\mathrm{Res}_t(j, j_t)$.*

5 Conclusions

In this paper we have presented an algorithm for computing a critical set of a family of rational curves depending on a parameter. From the critical set, the topology types in the family can be derived. The algorithm is based on a geometric interpretation of known results for the implicit case, and on advantages of parametric representation, and requires certain properties on the family to be analyzed. These properties can be algorithmically checked. In our experimentation, we have found that the timings of the parametric algorithm are usually quite better than those of the implicit algorithm; in fact, the parametric algorithm is able to manage inputs that the implicit algorithm cannot deal with. On the other hand, the drawback of the provided algorithm is that it may determine critical sets bigger than those determined by the implicit algorithm, therefore containing superfluous values with respect to the implicit critical set. So, as a potential future line of research, one could address the problem of reducing the size of the output, trying to approach optimality. Furthermore, the method applies with exact coefficients. So, it would also be nice to consider the (challenging) possibility of applying it in the case of approximate coefficients.

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6 Appendix I: Parametrizations of the families used in the comparison table.

Family 1:

$$u := -78t^4 + 62t^3 + 11t^2 + 88t + 1 + \lambda(30t^4 + 81t^3 - 5t^2 - 28t + 4)$$
$$v := -11t^4 + 10t^3 + 57t^2 - 82t - 48 + \lambda(-11t^4 + 38t^3 - 7t^2 + 58t - 94)$$

The implicit equation has degree 4 (as a polynomial in x, y) and 95 terms; the total degree can be found (for this family and for the rest of the families in this appendix) in the comparison table provided in Subsection 4.

Family 2:

$$u := 50 - 85t^5 - 55t^4 - 37t^3 - 35t^2 + 97t + \lambda(-59 + 79t^5 + 56t^4 + 49t^3 + 63t^2 + 57t)$$
$$v := -62 + 45t^5 - 8t^4 - 93t^3 + 92t^2 + 43t + \lambda(-61 + 77t^5 + 66t^4 + 54t^3 - 5t^2 + 99t)$$

The implicit equation has degree 5 (as a polynomial in x, y) and 161 terms.

Family 3:

$$u := \frac{2t^7 + t^5\lambda + \lambda^2 t^4 - 3t^3\lambda^2 + 3\lambda^3 t^2 - t^3 + 3t\lambda - 2\lambda^2 - 2t^4\lambda + t^6\lambda - t^4}{(t^4 - 2t\lambda + \lambda^2)(t^3 - t\lambda + \lambda^2)}$$
$$v := \frac{t^3 + \lambda t^2 - 1}{t^3 - t\lambda + \lambda^2}$$

The implicit equation has degree 7 (as a polynomial in x, y) and 343 terms.

Family 4:

$$u := \frac{-35 - 85t^3 - 55t^2 - 37t}{56 + 97t^3 + 50t^2 + 79t} + \lambda \frac{66 + 43t^3 - 62t^2 + 77t}{-61 + 54t^3 - 5t^2 + 99t}$$

$$v := \frac{31 - 50t^3 - 12t^2 - 18t}{56 + 97t^3 + 50t^2 + 79t} + \lambda \frac{-59 + 49t^3 + 63t^2 + 57t}{-61 + 54t^3 - 5t^2 + 99t}$$

The implicit equation has degree 6 (as a polynomial in x, y) and 84 terms.

Family 5:

$$u := 50 - 85t^5 - 55t^4 - 37t^3 - 35t^2 + 97t + \lambda(-59 + 79t^5 + 56t^4 + 49t^3 + 63t^2 + 57t)$$

$$v := -62 + 45t^5 - 8t^4 - 93t^3 + 92t^2 + 43t + \lambda(-61 + 77t^5 + 66t^4 + 54t^3 - 5t^2 + 99t)$$

The implicit equation has degree 5 (as a polynomial in x, y) and 161 terms.

Family 6:

$$u := -47t - 91\lambda^2 - 47t^3 - 61\lambda^4 + 41t^5 - 58t^2\lambda^3$$

$$v := 23t^2 - 84t^3\lambda + 19t^2\lambda^2 - 50t\lambda^3 + 88t^5\lambda - 53t^2\lambda^4$$

The implicit equation has degree 5 (as a polynomial in x, y) and 191 terms.

Family 7:

$$u := \frac{t^5 - t^2\lambda^2 - t - 2\lambda + 1}{t^3 - t^2 + t\lambda - \lambda^2}$$

$$v := \frac{t^5 + t^2\lambda - t - 2\lambda^2 + 1}{t^3 - t^2 + t\lambda - \lambda^2}$$

The implicit equation has degree 5 (as a polynomial in x, y) and 219 terms.

Family 8:

$$u := \frac{-7t + 58t^2 - 94t\lambda - 68t^3 + 14t^2\lambda - 35\lambda^3}{-14 - 9t - 51\lambda - 73t^2 - 73t\lambda - 91\lambda^2}$$

$$v := \frac{-50 + 50\lambda + 67t^2 - 39t\lambda + 8\lambda^2 - 49t\lambda^2}{-14 - 9t - 51\lambda - 73t^2 - 73t\lambda - 91\lambda^2}$$

The implicit equation has degree 3 (as a polynomial in x, y) and 72 terms.

Family 9:

$$u := \frac{-5 + 99t - 61\lambda - 50\lambda^3 - 12t^6 - 18\lambda^6}{31 - 26t - 62\lambda + t^2 - 47t\lambda - 91\lambda^2}$$
$$v := -1 + 94t^2 + 83\lambda^2 - 86t\lambda^2 + 23\lambda^3 - 84t^3\lambda$$

The implicit equation has degree 6 (as a polynomial in x, y) and 204 terms.

Family 10:

$$u := \frac{-85 - 55t - 37\lambda - 35t^2 + 97t\lambda + 50\lambda^2}{79 + 56t + 49\lambda + 63t^2 + 57t\lambda - 59\lambda^2}$$
$$v := \frac{45 - 8t - 93\lambda + 92t^2 + 43t\lambda - 62\lambda^2}{79 + 56t + 49\lambda + 63t^2 + 57t\lambda - 59\lambda^2}$$

The implicit equation has degree 2 (as a polynomial in x, y) and 30 terms.

Family 11:

$$u := \frac{97\lambda + 50t\lambda + 79\lambda^2 + 56t^3 + 49t\lambda^2 + 63\lambda^3}{-93t + 92\lambda + 43t\lambda - 62t^3 + 77t\lambda^2 + 66\lambda^3}$$
$$v := \frac{-12 - 18t + 31\lambda - 26t\lambda - 62\lambda^2 + t^2\lambda}{-93t + 92\lambda + 43t\lambda - 62t^3 + 77t\lambda^2 + 66\lambda^3}$$

The implicit equation has degree 3 (as a polynomial in x, y) and 97 terms.

Family 12:

$$u := \frac{57t - 59t\lambda + 45\lambda^2 - 8t^3 - 93t\lambda^2 + 92t^2\lambda^2}{-18t + 31t^2 - 26t\lambda - 62t^3 + t^2\lambda^2 - 47\lambda^4}$$
$$v := \frac{-1 + 94t^2 + 83\lambda^2 - 86t\lambda^2 + 23\lambda^3 - 84t^3\lambda}{-18t + 31t^2 - 26t\lambda - 62t^3 + t^2\lambda^2 - 47\lambda^4}$$

The implicit equation has degree 3 (as a polynomial in x, y) and 146 terms.